Completing the Market in a Trinomial Model

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Abstract

This report aims to determine the no-arbitrage bounds for the price of options and discusses a method called completing the market in both the one-period and multi-period trinomial model. More precisely, in a trinomial model, we can use linear programming to find out the no-arbitrage bounds for the price of the options, the upper bound of the European call option with strike price $K = S_0m$ is $S_0(u-m)(R-d)$ and the lower bound of the option price will vary with the range of $R$. When $m < R < u$, the lower bound of the option price is $S_0(1 - \frac{m}{R})$, when $d < R \leq m$ the lower bound of the option price is 0. Specially, for the one-period trinomial model, we can add a call with strike to the primary assets. With this additional asset each claim can be replicated if and only if $S_0d < K < S_0u$. However, options cannot be replicated in this way when using the two-period trinomial model.

Keywords

Market completion; Replicating portfolio; Trinomial model.

1. Introduction

The pricing model of binomial tree option is firstly proposed by Cox, Ross and Rubinstein, how to use the tree discrete structure to model, explaining the price movement of the underlying asset in the quasi-discontinuous time case [1].

Consider that the binomial tree model assumes that at each node the stock price only goes up and down, and in fact the stock price does not change significantly in most case. The trinomial model adds a constant state of stock price at each node. [2] The trinomial tree is a lattice based computational model used in financial mathematics to price options. It was developed by Phelim Boyle in 1986. It is an extension of the binomial options pricing model, and is conceptually similar [3]. In the trinomial model, the stock price can go up, down or stay stable by factors $u$, $d$ or $m$ respectively. For example, in the one-period trinomial model, the stock price at time 1 has three possible values, or, where $S_0$ is the initial stock price.

On that basis, Tian obtained risk neutral probability by using the trinomial model method, but expressed as a complex exponential function of time [4], Guo Zijun and Zhang Chaoqing used the no-arbitrage equilibrium analysis technique to study the pricing problem of asset options under the trinomial model. [5]

In binomial models, options can be replicated by a portfolio consisting of one riskless asset and one stock. Under the assumption of no-arbitrage, the replicating portfolio can be used to calculate the price of options. However, replicating an option using two primary assets is not always possible in a trinomial model. In this case, we use linear programming to figure out the no-arbitrage bounds for the price. Moreover, if a third asset can be added to the replicating
portfolio then sometimes it is possible to replicate the option. We study how this can be done in this paper.

Table 1. The symbol used in the report

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>K</td>
<td>The strike price of the European call option</td>
<td></td>
</tr>
<tr>
<td>$S_t$</td>
<td>The price of the stock at time</td>
<td></td>
</tr>
<tr>
<td>$C_t$</td>
<td>The price of the European call option at time</td>
<td></td>
</tr>
<tr>
<td>$N^B$</td>
<td>The number of the risk-free asset</td>
<td></td>
</tr>
<tr>
<td>$N^S$</td>
<td>The number of the risky asset</td>
<td></td>
</tr>
<tr>
<td>$N^C$</td>
<td>The number of the call option</td>
<td></td>
</tr>
<tr>
<td>$F(S_t)$</td>
<td>The payoff of an option at time</td>
<td></td>
</tr>
<tr>
<td>$R$</td>
<td>Constant (return on the risk-free asset)</td>
<td>$R&gt;1$</td>
</tr>
<tr>
<td>$u, m, d$</td>
<td>Constants</td>
<td>$d &lt; m &lt; u$</td>
</tr>
<tr>
<td>$q_u, q_m, q_d$</td>
<td>Risk-neutral probabilities</td>
<td>$0 \leq q_u, q_m, q_d \leq 1$</td>
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</table>

2. Characterization of Replicable Claims

In this section, we introduce one general theorem about replicable claims and then discuss an example where the option cannot be replicated by two primary assets. In that example, we use the concept of linear programming.

Theorem 2.1:

An option with payoff $F$ can be replicated by riskless assets and stocks if and only if the equality

$$(u - m)F(S_0d) + (m - d)F(S_0u) = (u - d)F(S_0m) \quad (*)$$

holds.

Proof:

Consider a replicating portfolio consisting of $N^B$ units of bonds and $N^S$ shares of stock; i.e.,

$$N^B R + N^S S_0 u = F(S_0 u) \quad (1)$$
$$N^B R + N^S S_0 m = F(S_0 m) \quad (2)$$
$$N^B R + N^S S_0 d = F(S_0 d) \quad (3)$$

Need to show that equations (1), (2), (3) can be solved simultaneously if and only if (*) holds. First look at equations (1) and (3),

$$(1) - (3) : N^S S_0 (u - d) = F(S_0 u) - F(S_0 d)$$
$$\Rightarrow N^S = \frac{F(S_0 u) - F(S_0 d)}{S_0 (u - d)}.$$

Substitute $N^S$ into (3) to get

$$N^B = \frac{u F(S_0 d) - d F(S_0 u)}{R (u - d)}.$$

Now, equations (1), (2), (3) can be solved simultaneously if and only if $N^B$ and $N^S$ can also solve equation (2).

I.e., these equations can be solved if and only if

$$\frac{u F(S_0 d) - d F(S_0 u)}{(u - d)} + \frac{F(S_0 u) - F(S_0 d)}{(u - d)} \times m = F(S_0 m)$$
$$\Leftrightarrow (u - m)F(S_0 d) + (m - d)F(S_0 u) = (u - d)F(S_0 m).$$

This proves the theorem.

Now, we discuss an example where an option cannot be replicated by the two primary assets.
Example 2.1:
Consider a European call option with strike price $K = S_0m$. At time 1, the payoff of this option is
\[
F(S_1) = \begin{cases} 
S_0u - S_0m & \text{if } S_1 = S_0u \\
0 & \text{if } S_1 = S_0m \\
0 & \text{if } S_1 = S_0d.
\end{cases}
\]
Hence, it cannot be replicated by two primary assets as it does not satisfy (*). However, no-arbitrage price bounds can be found.
The lower bound for the price of this European call option can be determined by considering the following inequalities:
\[
\begin{align*}
N^B R + N^S S_0 u & \leq S_0u - S_0m \\
N^B R + N^S S_0 m & \leq 0 \\
N^B R + N^S S_0 d & \leq 0.
\end{align*}
\]
By the no-arbitrage principle, the price of this option at time 0 must be greater than or equal to the initial value of the replicating portfolio; i.e., $F_0 \geq N^B + N^S S_0$. Hence, if a maximum of $N^B + N^S S_0$ can be determined then a lower bound of $F_0$ can be obtained.
A range of $N^B$ and $N^S$ which satisfy the three inequalities can be found. Also, this can be considered as a linear programming problem which can be solved graphically.
The constraints are:
\[
N^S \leq \frac{R}{S_0 u} N^B + 1 - \frac{m}{u} \\
N^S \leq \frac{R}{S_0 m} N^B \\
N^S \leq \frac{R}{S_0 d} N^B.
\]
The objective function is:
\[
N^B + N^S S_0 = k \\
\Rightarrow N^S = -\frac{1}{S_0} N^B + c.
\]
In this case, a critical point which can maximise $N^B + N^S S_0$ is needed.

Figure 1. The corresponding critical region

If $m < R < u$, the maximum is the intersection of the equations:
\[
N^S = -\frac{R}{S_0 u} N^B + 1 - \frac{m}{u} \quad \text{and} \quad N^S = -\frac{R}{S_0 m} N^B.
\]
I.e.,
Substitute $N^B$ and $N^S$ into $N^B + N^S S_0$ to get a lower bound for the option price, which is:

$$1 - \frac{m}{u} + S_0 = S_0 \left(1 - \frac{m}{R}\right).$$

If $d < R \leq m$, the maximum occurs at $(0,0)$, thus the lower bound for the option price is 0.

Similarly, the upper bound for the price of this European call option can be determined by considering:

$$N^B R + N^S S_0 u \geq S_0 u - S_0 m,$$

$$N^B R + N^S S_0 m \geq 0,$$

$$N^B R + N^S S_0 d \geq 0.$$

Thus, the coordinates of the intersection of $N^S = -\frac{R}{S_0 u} N^B + 1 - \frac{m}{u}$ and $N^S = -\frac{R}{S_0 d} N^B$ to get

$$N^B = -\frac{S_0 d (u - m)}{R (u - d)},$$

$$N^S = \frac{u - m}{u - d}. $$

Thus,

$$N^B + N^S S_0 \geq \frac{S_0 d (u - m)}{R (u - d)} + \frac{u - m}{u - d} S_0 = \frac{S_0 (u - m) (R - d)}{R (u - d)}.$$
From the previous result we get

\[ F_0 \leq N^B + N^S S_0. \]

Therefore,

\[ \frac{S_0(u - m)(R - d)}{R(u - d)} \]

is an upper bound for \( F_0 \).

From what has been discussed above, we mainly use linear programming to figure out that the upper bound of the European call option with strike price \( K = S_0 m \) is \( \frac{S_0(u - m)(R - d)}{R(u - d)} \), and the lower bound of the option price will vary with the range of \( R \). When \( m < R < u \), the lower bound of the option price is \( S_0 (1 - \frac{m}{R}) \) and when \( d < R \leq m \) the lower bound of the option price is 0.

However, when solving more general problems using linear programming, the graph cannot be determined precisely. Thus, linear programming might be very difficult to apply to general problems.

3. Market Completion

In economics, a complete market is a market with two conditions:

1. Negligible transaction costs and therefore also perfect information
2. there is a price for every asset in every possible state of the world [6]

Now, use the money market, stock and a call option with the given price \( C_0 \) to find the price of another contingent claim. This method is called ‘completing the market’.

First, form a replicating portfolio consisting of \( N^B \) units of bonds, \( N^S \) shares of stock and \( N^C \) units of call option. Denote the payoff of the call option by \( G^C \).

Thus, the following equations hold:

\[
\begin{align*}
N^B R + N^S S_0 u + N^C G^C(S_0 u) &= F(S_0 u) \\
N^B R + N^S S_0 m + N^C G^C(S_0 m) &= F(S_0 m) \\
N^B R + N^S S_0 d + N^C G^C(S_0 d) &= F(S_0 d).
\end{align*}
\]

Example 3.1:

Consider a specific case with strike price \( K = S_0 m \):

\[
\begin{align*}
N^B R + N^S S_0 u + N^C S_0(u - m) &= F(S_0 u) \quad (1) \\
N^B R + N^S S_0 m + N^C \times 0 &= F(S_0 m) \quad (2) \\
N^B R + N^S S_0 d + N^C \times 0 &= F(S_0 d) \quad (3)
\end{align*}
\]

We now solve these equations as follows:

\[
(2) - (3) : N^S S_0(m - d) = F(S_0 m) - F(S_0 d) \Rightarrow N^S = \frac{F(S_0 m) - F(S_0 d)}{S_0(m - d)}
\]

Substitute \( N^S \) into (3) to get

\[
N^B = \frac{m F(S_0 d) - d F(S_0 m)}{R(m - d)}.
\]

Then substitute \( N^B \) and \( N^S \) into \( N^B R + N^S S_0 u + N^C S_0(u - m) = F(S_0 u) \) to obtain

\[
\frac{m F(S_0 d) - d F(S_0 m)}{R(m - d)} R + \frac{F(S_0 m) - F(S_0 d)}{S_0(m - d)} S_0 u + N^C S_0(u - m) = F(S_0 u) \Rightarrow N^C = \frac{F(S_0 u)(m - d) - F(S_0 m)(u - d) + F(S_0 d)(u - m)}{S_0(u - m)(m - d)}.
\]
By no-arbitrage, the initial price \( F_0 \) must be the same as the initial value of the replicating portfolio, i.e.,

\[
\frac{m F(S_0 d) - d F(S_0 m)}{R (m - d)} R + \frac{F(S_0 m) - F(S_0 d)}{S_0 (m - d)} S_0 u + NC S_0 (u - m) = F(S_0 u)
\]

\[
\Rightarrow NC = \frac{F(S_0 u)(m - d) - F(S_0 m)(u - d) + F(S_0 d)(u - m)}{S_0 (u - m)(m - d)}.
\]

Recall that the no-arbitrage bounds for \( C_0 \) have been determined before using linear programming, thus can be substituted into the above formula to obtain the bounds for \( F_0 \) as follows.

If \( d < R \leq m \):

\[
0 \leq C_0 \leq \frac{S_0 (u - m)(R - d)}{R (u - d)}.
\]

If \( m < R < u \):

\[
S_0 \left(1 - \frac{m}{R}\right) \leq C_0 \leq \frac{S_0 (u - m)(R - d)}{R (u - d)}.
\]

Therefore,

If \( d < R \leq m \) and \( F(S_0 u)(m - d) - F(S_0 m)(u - d) + F(S_0 d)(u - m) \geq 0 \),

The lower bound for \( F_0 \) is

\[
\frac{m F(S_0 d) - d F(S_0 m)}{R (m - d)} + \frac{F(S_0 m) - F(S_0 d)}{m - d}
\]

and the upper bound for \( F_0 \) is

\[
\frac{m F(S_0 d) - d F(S_0 m)}{R (m - d)} + \frac{F(S_0 m) - F(S_0 d)}{m - d} + \frac{(R - d)[F(S_0 u)(m - d) - F(S_0 m)(u - d) + F(S_0 d)(u - m)]}{R (m - d)(u - d)}.
\]

If \( m < R < u \) and \( F(S_0 u)(m - d) - F(S_0 m)(u - d) + F(S_0 d)(u - m) \geq 0 \),

The lower bound for \( F_0 \) is

\[
\frac{m F(S_0 d) - d F(S_0 m)}{R (m - d)} + \frac{F(S_0 m) - F(S_0 d)}{m - d} + \frac{(R - m)[F(S_0 u)(m - d) - F(S_0 m)(u - d) + F(S_0 d)(u - m)]}{R (m - d)(u - m)}.
\]

and the upper bound for \( F_0 \) is

\[
\frac{m F(S_0 d) - d F(S_0 m)}{R (m - d)} + \frac{F(S_0 m) - F(S_0 d)}{m - d} + \frac{(R - d)[F(S_0 u)(m - d) - F(S_0 m)(u - d) + F(S_0 d)(u - m)]}{R (m - d)(u - d)}.
\]

If \( d < R \leq m \) and \( F(S_0 u)(m - d) - F(S_0 m)(u - d) + F(S_0 d)(u - m) < 0 \),

the lower bound for \( F_0 \) is

\[
\frac{m F(S_0 d) - d F(S_0 m)}{R (m - d)} + \frac{F(S_0 m) - F(S_0 d)}{m - d} + \frac{(R - d)[F(S_0 u)(m - d) - F(S_0 m)(u - d) + F(S_0 d)(u - m)]}{R (m - d)(u - d)}
\]

and the upper bound for \( F_0 \) is

\[
\frac{m F(S_0 d) - d F(S_0 m)}{R (m - d)} + \frac{F(S_0 m) - F(S_0 d)}{m - d}.
\]

If \( m < R < u \) and \( F(S_0 u)(m - d) - F(S_0 m)(u - d) + F(S_0 d)(u - m) < 0 \),

\[
\frac{m F(S_0 d) - d F(S_0 m)}{R (m - d)} + \frac{F(S_0 m) - F(S_0 d)}{m - d}.
\]
the lower bound for $F_0$ is
\[
\frac{mF(S_0d) - dF(S_0m)}{R(m - d)} + \frac{F(S_0m) - F(S_0d)}{R(m - d)}
\]
and the upper bound for $F_0$ is
\[
\frac{mF(S_0d) - dF(S_0m)}{R(m - d)} + \frac{F(S_0m) - F(S_0d)}{R(m - d)} + \frac{(R - m)F(S_0u)(m - d) - F(S_0m)(u - d) + F(S_0d)(u - m)}{R(m - d)(u - m)}.
\]

Note that if the option can be replicated by two primary assets, i.e., $F(S_0u)(m - d) - F(S_0m)(u - d) + F(S_0d)(u - m) = 0$ (by Theorem 2.1) then the upper bound and the lower bound will be the same.

**Theorem 3.1:**

Suppose that a European call option has payoff $F$ and strike price $K = S_0 m$, then the no-arbitrage price $F_0$ is
\[
F_0 = \frac{1}{R} \left\{ q_u F(S_0u) + q_m F(S_0m) + q_d F(S_0d) \right\},
\]

Where
\[
q_u = \frac{C_0R}{S_0(u - m)}; \quad q_m = \frac{S_0(u - m)(R - d) + C_0R(d - u)}{S_0(m - d)(u - m)}; \quad q_d = \frac{S_0(u - m)(m - R) + C_0R(u - m)}{S_0(m - d)(u - m)}.
\]

**Proof:**

From Example 3.1,
\[
F = \frac{S_0(u - m)(R - d)F(S_0m) + S_0(u - m)(m - R)F(S_0d)}{S_0(m - d)(u - m)F(S_0u) + S_0(u - m)(R - d) + C_0R(d - u)} + \frac{F(S_0m)(d - u) + F(S_0d)(u - m)}{S_0(m - d)(u - m)}.
\]

It follows that the no-arbitrage price $F_0$ is
\[
F_0 = \frac{1}{R} \left\{ q_u F(S_0u) + q_m F(S_0m) + q_d F(S_0d) \right\},
\]

Where
\[
q_u = \frac{C_0R(m - d)}{S_0(m - d)(u - m)} = \frac{C_0R}{S_0(u - m)}; \quad q_m = \frac{S_0(u - m)(R - d) + C_0R(d - u)}{S_0(m - d)(u - m)}; \quad q_d = \frac{S_0(u - m)(m - R) + C_0R(u - m)}{S_0(m - d)(u - m)}.
\]

Note that $q_u, q_m$ and $q_d$ satisfy the condition $q_u + q_m + q_d = 1$. This proves the theorem.

Next we consider the general case $K \neq S_0 m$ and work out whether the corresponding call can be used to 'complete the market'.

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Theorem 3.2:
A call option with payoff $G$ can be used to complete the market if and only if the strike price $K$ satisfies $S_0 d < K < S_0 u$.

Proof:
Let $F$ denote the payoff of an arbitrary option.

If $S_0 m < K < S_0 u$, the following hold:
\[
\begin{align*}
N^B R + N^S S_0 u + N^C (S_0 u - K) &= F(S_0 u) \\
N^B R + N^S S_0 m &= F(S_0 m) \\
N^B R + N^S S_0 d &= F(S_0 d).
\end{align*}
\]

This is similar to the case with $K = S_0 m$ so $N^B$ and $N^S$ and $N^C$ can be found easily.

If $K \geq S_0 u$, the payoff $G$ is always 0 thus the equations become
\[
\begin{align*}
N^B R + N^S S_0 u &= F(S_0 u) \\
N^B R + N^S S_0 m &= F(S_0 m) \\
N^B R + N^S S_0 d &= F(S_0 d).
\end{align*}
\]

Now, there are 3 equations but only 2 unknowns and the value of $N^C$ cannot be determined. Therefore, any payoff that does not satisfy (*) cannot be replicated by these assets; see also Theorem 2.1.

If $S_0 d < K < S_0 m$, the equations are
\[
\begin{align*}
N^B R + N^S S_0 u + N^C (S_0 u - K) &= F(S_0 u) \\
N^B R + N^S S_0 m + N^C (S_0 m - K) &= F(S_0 m) \\
N^B R + N^S S_0 d &= F(S_0 d).
\end{align*}
\]

The equations can be solved if and only if the coefficient matrix $A$ and the augmented $(A|d)$ have the same rank.

The rank of the coefficient matrix $A$ can be found as follows
\[
A = \begin{bmatrix}
R & S_0 u & S_0 u - K \\
R & S_0 m & S_0 m - K \\
R & S_0 d & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
R & S_0 u & S_0 u - K \\
0 & 1 & 1 \\
0 & 0 & K - S_0 d
\end{bmatrix}
\]

Thus, the rank of $A$ is 3.

Similarly, we can figure out the rank of the augmented matrix $(A|d)$.
\[
(A|d) = \begin{bmatrix}
R & S_0 u & S_0 u - K & F(S_0 u) \\
R & S_0 m & S_0 m - K & F(S_0 m) \\
R & S_0 d & 0 & F(S_0 d) \\
0 & S_0 (m - u) & S_0 (m - u) & F(S_0 m) - F(S_0 u) \\
0 & 0 & K - S_0 d & F(S_0 d) - F(S_0 u) - \frac{F(S_0 m) - F(S_0 u)}{m - u}
\end{bmatrix}
\]

Thus, the equations can be solved if and only if the rank of $(A|d)$ is 3 which is equal to the rank of $A$.

Therefore, if $S_0 m < K < S_0 u$ the equations can be solved and the option with payoff $F$ can be replicated by 3 assets.

If $K \leq S_0 d$, the payoff $G$ of the call is
\[
G(S_1) = \begin{cases}
S_0 u - K & \text{if } S_1 = S_0 u \\
S_0 m - K & \text{if } S_1 = S_0 m \\
S_0 d - K & \text{if } S_1 = S_0 d.
\end{cases}
\]

Hence, the following hold:
Let $x = N^B R - N^C K$ and $y = N^C + N^S$. we get
\[
\begin{align*}
N^B R + N^S S_0 u + N^C (S_0 u - K) &= F(S_0 u) \\
N^B R + N^S S_0 m + N^C (S_0 m - K) &= F(S_0 m) \\
N^B R + N^S S_0 d + N^C (S_0 d - K) &= F(S_0 d).
\end{align*}
\]

Again, the number of unknowns is less than the number of equations so the solutions may not exist.

Overall, the call option with payoff $G$ can only be used in a portfolio to replicate another option with payoff $F$ if and only if the strike price $K$ satisfies $S_0 d < K < S_0 u$.

This proves the theorem.

4. Two-Period Trinomial Model

Corollary to Theorem 3.2:

In a two-period trinomial model, options can not be replicated by a portfolio consisting of bonds, stocks and call options.

Proof:

For sake of simplicity, assume that $m^2 = ud$.

Suppose that option with payoff $F$ can be replicated by a portfolio consisting of bonds, stocks and another call option.

At time 1, we need to consider three sets of equations at the same time.
\[
\begin{align*}
N^B R + N^S S_0 u^2 + N^C (S_0 u^2 - K, 0^+) &= F(S_0 u^2) \\
N^B R + N^S S_0 um + N^C (S_0 um - K, 0^+) &= F(S_0 um) \\
N^B R + N^S S_0 ud + N^C (S_0 ud - K, 0^+) &= F(S_0 ud). 
\end{align*}
\]
\[
\begin{align*}
N^B R + N^S S_0 um + N^C (S_0 um - K, 0^+) &= F(S_0 um) \\
N^B R + N^S S_0 ud + N^C (S_0 ud - K, 0^+) &= F(S_0 ud) \\
N^B R + N^S S_0 md + N^C (S_0 md - K, 0^+) &= F(S_0 md). 
\end{align*}
\]
\[
\begin{align*}
N^B R + N^S S_0 ud + N^C (S_0 ud - K, 0^+) &= F(S_0 ud) \\
N^B R + N^S S_0 md + N^C (S_0 md - K, 0^+) &= F(S_0 md) \\
N^B R + N^S S_0 d^2 + N^C (S_0 d^2 - K, 0^+) &= F(S_0 d^2). 
\end{align*}
\]

From Theorem 3.2,
(a) implies that $S_0 ud < K < S_0 u^2$;
(b) implies that $S_0 md < K < S_0 um$;
(c) implies that $S_0 d^2 < K < S_0 ud$.

However, these three inequalities can not be satisfied simultaneously.

This is a contradiction and thus proves the corollary.
5. Conclusion

In conclusion, according to the linear programming process, the upper bound of the European call option with strike price $K = S_0m$ is $S_0\frac{(u-m)(R-d)}{R(u-d)}$, and the lower bound of the option price will vary with the range of $R$. When $m < R < u$, the lower bound of the option price is $S_0 \left(1 - \frac{m}{R}\right)$, when $d < R \leq m$, the lower bound of the option price is $0$. Specially, for the one-period trinomial model, the work can add a call with strike $K$ to the primary assets. With this additional asset each claim can be replicated if and only if $S_0d < K < S_0u$. However, options cannot be replicated in this way when using the two-period trinomial model. Therefore, alternative method should be determined to tackle this problem.

References


